Indian Statistical Institute, Bangalore
M. Math.
Second Year, Second Semester
Advanced Functional Analysis

Final Examination — Date : May 19, 2021
Maximum marks: 100 — Time: 3 hours
Instructor: B V Rajarama Bhat

Notation: In the following $\mathcal{H}$ is a complex separable Hilbert space and $B(\mathcal{H})$ denotes the algebra of all bounded operators on $\mathcal{H}$.

(1) Suppose A, B are bounded operators on $\mathcal{H}$ and $\{A_n\}_{n\in\mathbb{N}}$, $\{B_n\}_{n\in\mathbb{N}}$ are two sequences of bounded operators on $\mathcal{H}$. (i) Show that if $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ converge in SOT to A, B respectively, then $\{A_nB_n\}_{n\in\mathbb{N}}$ converges to AB in SOT. (ii) Give an example where $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ converge in WOT to A, B respectively, but $\{A_nB_n\}_{n\in\mathbb{N}}$ does not converge to AB in WOT. [15]

(2) Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$. Let $V : \mathcal{H} \to \mathcal{H}$ be the unilateral shift defined by $Ve_n = e_{n+1}, n \in \mathbb{N}$, and extended linearly and continuously. Take $R = 2I + V + V^*$. (i) Show that R is a positive operator. (ii) Let E be the spectral measure of R. Compute first three moments of the probability measure E_{e_1,e_1}. [15]

(3) Fix a natural number n. Let $M = \{(A_1, A_2, \ldots, A_n) : A_j \in B(\mathcal{H}), 0 \leq A_j \leq I, \forall j\}$, considered as a convex subset of the vector space $B(\mathcal{H})^n = \{(X_1, X_2, \ldots X_n) : X_j \in B(\mathcal{H})\}$. (i) Show that if $P_1, P_2, \ldots, P_n$ are projections such that $P_1+P_2+\cdots+P_n = I$, then they are mutually orthogonal and $(P_1, P_2, \ldots, P_n)$ is an extreme point of M. (ii) If $n = 2$, then show that the converse is also true, namely that if (R_1, R_2) are extreme points of M, then R_1, R_2 are projections satisfying $R_1 + R_2 = I$. [15]

(4) Let $N = UP$ be the polar decomposition of a bounded operator N on $\mathcal{H}$. Show that N is normal if and only if U and P commute. [15]

(5) Let B be a bounded normal operator on $\mathcal{H}$. Show that there exists a bounded normal operator A such that $B = A^2$. [15]

(6) Let E, F are projections in a von Neumann algebra $\mathcal{A}$. Show that (i) $(E \vee F - F) \sim E - E \wedge F$; (ii) $(E - E \wedge F^{\perp}) \sim (F - E^{\perp} \wedge F)$. [15]

(7) Consider group von Neumann algebras $\mathcal{L}(\mathbb{Z})$ and $\mathcal{R}(\mathbb{Z})$ for the group of integers $\mathbb{Z}$. (i) Show that $\mathcal{L}(\mathbb{Z}) = \mathcal{R}(\mathbb{Z})$ and is an abelian von Neumann algebra. (ii) Let τ be the associated tracial state on $\mathcal{L}(\mathbb{Z})$ coming from unit vector δ_0. Show that for any $0 \leq s \leq 1$, there exists a projection E in $\mathcal{L}(\mathbb{Z})$ such that

$$\tau(E) = s.$$

(Hint: Use the Hilbert space isomorphism between $l^2(\mathbb{Z})$ and $L^2(\mathbb{T})$, where $\mathbb{T}$ is the circle $\{e^{2\pi it}, 0 \leq t \leq 1\}$ given by the standard basis vectors e_n mapping to functions

$$f_n(t) = e^{2\pi it},\ 0 \leq t \leq 1$$

for $n \in \mathbb{Z}$.) [15]